*Asian Journal of Statistical Sciences*
Vol. 2, No-1, (2022), pp. 47-60


ARF INDIA
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Journal homepage link:
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# Estimation of the parameters of the generalized inverted Kumaraswamy distribution under the first failure-censored sampling plan

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**ARTICLE HISTORY**
Compiled August 18, 2022

Received 02 May 2022; Accepted 20 June 2022

**ABSTRACT**
In this paper we develop approximate Bayes estimators of the shape parameters of the generalized inverted Kumaraswamy (GIKum) distribution based on the progressive first-failure censored plan. We consider the maximum likelihood and Bayesian estimations with gamma-informative prior distribution for the parameters, reliability function, hazard rate and reversed hazard rate functions. We apply the Lindley's approximation and Markov Chain Monte Carlo (MCMC) methods. The Bayes estimators have been obtained relative to both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. Finally, to assess the performance of the proposed estimators, some numerical results using simulation study concerning different sample sizes are given.

**KEYWORDS**
Generalized inverted Kumaraswamy distribution; progressive first-failure censored; loss functions; Lindley's approximation
**2010 AMS Classification**: 62F15, 62N01, 62N02, 62N05

## 1. Introduction

Life testing experiments are usually time consuming and costly. We therefore, use various types of censoring schemes to cut short the experiment. The censoring scheme in an experiment may also arise naturally without the control of the experimenter. For example, in medical studies a patient may drop out of a study before its completion. Initially, the popular censoring schemes were conventional type I and type II. In some life tests, there may be an urgent need to use some test items that have not yet failed for other purposes before the end of the test, cf. Mann et al. (1974) and Sinha (1986). Cohen (1963) thought over this point and introduced progressive type II censoring scheme which allows removal of items from the experiment before the final termination point. Balakrishnan and Aggarwala (2000) compiled the work done on progressive censoring up-to year 1999. Progressive censoring has also been studied by many authors like Pradhan and Kundu (2009) and Krishna and Kumar (2013). There are situations

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in real life where lifetimes of items are very high and test facilities are limited. If the test material is comparatively cheaper, one can put $k \times n$ items on test instead of only $n$ units. In this case $n$ sets or groups each consisting of $k$ items are put on test separately. In each set only first-failure is observed and the progressive censoring is applied to $n$ groups. Johnson (1964) studied this type of grouping of units and observing only first-failure. Some other studies of this type of grouping of units are by Balasooriya (1995), Wu et al. (2003), and Wu and Yu (2005). The combination of first-failure and progressive censoring is known as progressive first-failure censoring scheme. This concept was given by Wu and Kus (2009). They described estimation methods in the case of a Weibull distribution using this new censoring plan. More recent references can be found in Lio and Tsai (2012), Kumar et al. (2015) and Dube et al. (2016). Now, we describe the progressive first-failure censoring scheme in more details. Assume that $k \times n$ items are put on test in $n$ independent groups with $k$ items in each group. Let us adopt a progressive censoring scheme $\underline{R} = (R_1, R_2, ..., R_m)$. Upon the first-failure of a unit, we remove that group in which first-failure occurred and $R_1$ additional groups randomly from the remaining $(n-1)$ groups in the experiment. As soon as the second failure takes place we remove that group and additional $R_2$ groups randomly from remaining $(n - R_1 - 2)$ groups and so on. This procedure continues till the $m$th failure occurs when the remaining $R_m$ groups and the group in which last failure took place are removed. Obviously, $\sum_{i=1}^{m} R_i + m = n$. Also, if $R_1 = R_2 = ... = R_m = 0$, the progressive first-failure censoring scheme reduces to first-failure censoring scheme and if $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$, it reduces to first-failure type II censoring scheme, a progressive type II censored scheme when $k = 1$. It is worth noting that the progressive first-failure censored scheme with a cumulative distribution function (cdf) $F(y)$ can be viewed as a progressive type II censored sample from a population with the cdf $1 - (1 - F(y))^k$. For this reason, results for the progressive type II censored scheme can be easily extended to progressive first-failure censored scheme. Therefore, progressive first-failure censoring is a generalization of progressive censoring. Obviously, although more items are used in the progressive first-failure censoring plan than in others, it has advantages in terms of reducing test time. Let $y_{1:m:n:k},\ y_{2:m:n:k}, \cdots,\ y_{m:m:n:k}$ be a progressive first-failure censored sample from a population with the probability density function (pdf) $f(.)$ and cdf $F(.)$ with progressive censoring scheme $\underline{R}$. For simplicity, let us denote $(y_{1:m:n:k},\ y_{2:m:n:k}, \cdots,\ y_{m:m:n:k})$ by $\underline{y} = (y_1,\ y_2, ...,\ y_m)$. On the basis of a progressive first-failure censored sample $\underline{y}$ the likelihood function is given by [see Balakrishnan and Aggarwala (2000) and Wu and Kus (2009)]

$$L(\underline{y}) = \tau k^m \prod_{i=1}^{m} f(y_i)[1 - F(y_i)]^{(k(R_i+1)-1)}, \tag{1.1}$$

where $\tau = n(n - 1 - R_1)...(n - R_1 - ... - R_{m-1} - m + 1)$.

In the recent times, there has been an increased interest in applying some inverted distributions to data applications in the areas of medical, economic and engineering sciences, lifetime analysis, finance and insurance.

Kumaraswamy (Kum) distribution defined on the interval (0,1) was introduced by Kumaraswamy (1980). It is similar to the Beta distribution, but much simpler to use especially in simulation studies since its pdf and cdf can be expressed in closed form, for more detail about this family of distributions, see Barakat et al. (2017).Abd Al-Fattah et al. (2016) derived the inverted Kumaraswamy (IKum) distribution from the distribution after some transformations. Moreover, Iqbal et al. (2017) derived the GIKum distribution by using a power transformation. The pdf and cdf of the GIKum distribution are respectively given by

$$f(y;\ \lambda,\ \eta,\ \kappa) = \ \lambda\ \eta\ \kappa\ y^{\kappa-1}\ (1+y^{\kappa})^{-(\lambda+1)}\ [1-(1+y^{\kappa})^{-\lambda}]^{\eta-1}\ ,\ y>0,\ \lambda,\ \eta,\ \kappa>0 \tag{1.2}$$

and

$$F(y;\ \lambda,\ \eta,\ \kappa) = \ [1-(1+y^{\kappa})^{-\eta}]^{\lambda}\ ,\ y>0,\ \lambda,\ \eta,\ \kappa>0. \tag{1.3}$$

The main objective of this paper is to estimate the parameters of the GIKum distribution by Bayes estimators. Both the maximum likelihood estimation (MLE) and Bayesian methods are obtained based on progressive first-failure censoring schemes. The paper's organization is as follows: Section 2 deals with the MLE of the unknown parameters, as well as the functions of reliability, hazard rate and reversed hazard rate. We use Lindley's approximation, cf. Lindley (1980), for the calculation of Bayes estimates in Section 3. In Section 4, for comparisons of various estimates produced in this paper, a Monte Carlo simulation is performed. Concluding remarks are given in Section 5.

## 2. Maximum Likelihood Estimators

In this section, we derive the MLEs of the unknown parameters, reliability, hazard rate and reversed hazard rate functions, based on progressive first-failure censored samples. Assume that the failure time distribution is the GIKum distribution with the pdf and cdf defined in (1.2) and (1.3), respectively. From (1.1), (1.2) and (1.3), the likelihood function is given by

$$L(\underline{y};\ \lambda,\ \eta,\ \kappa) = \ \tau(k\lambda\eta\kappa)^{m}\ \prod_{i=1}^{m}(y_i^{\kappa-1}(1+y_i^{\kappa})^{-(\lambda+1)}[1-(1+y_i^{\kappa})^{-\lambda}]^{\eta-1})$$

$$\times\prod_{i=1}^{m}(1-[1-(1+y_i^{\kappa})^{-\eta}]^{\lambda})^{(k(R_i+1)-1)}. \tag{2.1}$$

The logarithm of the likelihood function may then be written as

$$\log L = \ \ell = \ \log\tau + m\ \log[k\lambda\eta\kappa] + (\kappa-1)\sum_{i=1}^{m}\log y_i - (\lambda+1)\sum_{i=1}^{m}\log[1+y_i^{\kappa}]$$

$$+(\eta-1)\sum_{i=1}^{m}\log[1-(1+y_i^{\kappa})^{-\lambda}]+\sum_{i=1}^{m}((k(R_i+1)-1)\log[1-(1-(1+y_i^{\kappa})^{-\eta})^{\lambda}]). \quad (2.2)$$

By calculating the first partial derivatives of (2.2) with respect to $\lambda$, $\eta$, and $\kappa$ and equating to zero, we obtain the likelihood equations

$$\left.\begin{aligned}
&\frac{m}{\lambda}-\sum_{i=1}^{m}\log\left[1+y_i^{\kappa}\right]+(\eta-1)\sum_{i=1}^{m}\frac{(1+y_i^{\kappa})^{-\lambda}\log\left[1+y_i^{\kappa}\right]}{1-(1+y_i^{\kappa})^{-\lambda}}\\
&=\sum_{i=1}^{m}\frac{(k\,(1+R_i)-1)\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}\log\left[1-(1+y_i^{\kappa})^{-\eta}\right]}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}},\\
&\frac{m}{\eta}+\sum_{i=1}^{m}\log\left[1-(1+y_i^{\kappa})^{-\lambda}\right]\\
&=\sum_{i=1}^{m}\frac{\lambda\,(k\,(1+R_i)-1)\,(1+y_i^{\kappa})^{-\eta}\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda-1}\log\left[1+y_i^{\kappa}\right]}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}},\\
&\frac{m}{\kappa}+\sum_{i=1}^{m}\log y_i-(1+\lambda)\sum_{i=1}^{m}\frac{y_i^{\kappa}\log y_i}{1+y_i^{\kappa}}+(\eta-1)\sum_{i=1}^{m}\frac{\lambda y_i^{\kappa}(1+y_i^{\kappa})^{-(\lambda+1)}\log y_i}{1-(1+y_i^{\kappa})^{-\lambda}}\\
&=\sum_{i=1}^{m}\frac{\eta\lambda\,(k\,(1+R_i)-1)\,y_i^{\kappa}(1+y_i^{\kappa})^{-(\eta+1)}\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda-1}\log y_i}{1-\left(1-(1+y_i^{\kappa})^{-\eta}\right)^{\lambda}}.
\end{aligned}\right\} \quad (2.3)$$

The solutions of the non-linear equations (2.3) are $\hat{\lambda}$, $\hat{\eta}$, and $\hat{\kappa}$. The MLEs of the reliability, hazard rate and reversed hazard rate functions are, respectively, given as

$$\hat{R}\,(t)=1-\left[1-\left(1+t^{\hat{\kappa}}\right)^{-\hat{\lambda}}\right]^{\hat{\eta}}, t>0,$$

$$\hat{H}\,(t)=\frac{\hat{\lambda}\ \hat{\eta}\ \hat{\kappa}\ t^{\hat{\kappa}-1}\left[1-\left(1+t^{\hat{\kappa}}\right)^{-\hat{\lambda}}\right]^{\hat{\eta}-1}}{(1+t^{\hat{\kappa}})^{\hat{\lambda}+1}\ (1-(1-(1+t^{\hat{\kappa}})^{-\hat{\lambda}})^{\hat{\eta}})}, t>0,$$

and

$$\hat{H}^{*}\,(t)=\frac{\hat{\lambda}\ \hat{\eta}\hat{\kappa}t^{\hat{\kappa}-1}}{(1+t^{\hat{\kappa}})^{\hat{\lambda}+1}\left[1-(1+t^{\hat{\kappa}})^{-\hat{\lambda}}\right]}, t>0.$$

## 3. Bayesian estimation

In this section, the Bayesian estimators of the unknown parameters $\lambda$, $\eta$, and $\kappa$ of the GIKum distribution are obtained. Also, we study the reliability, hazard rate and reversed hazard rate functions based on progressive first-failure censoring samples, under symmetric (squared error) and asymmetric (linex and general entropy) loss functions. Moreover the Lindley's approximation and MCMC methods are obtained. Assuming that $\lambda,\ \eta$, and $\kappa$ are independent random variables with gamma informative prior distributions respectively which are defined by

$$\pi_1(\lambda;\ \zeta_1,\ \nu_1) = \frac{e^{-\zeta_1\eta}{\zeta_1}^{\nu_1}}{\Gamma(\nu_1)}\ \lambda^{\nu_1-1};\ \ \lambda>0,\ (\zeta_1,\ \nu_1>0),$$

$$\pi_2(\eta;\ \zeta_2,\ \nu_2) = \frac{e^{-\zeta_2\eta}{\zeta_2}^{\nu_2}}{\Gamma(\nu_2)}\ \eta^{\nu_2-1};\ \ \eta>0,\ (\zeta_2,\ \nu_2>0),$$

and

$$\pi_3(\kappa;\ \zeta_3,\ \nu_3) = \frac{e^{-\zeta_3\kappa}{\zeta_3}^{\nu_3}}{\Gamma(\nu_3)}\ \kappa^{\nu_3-1};\ \ \kappa>0,\ (\zeta_3,\ \nu_3>0).$$

Then the joint prior distribution for $\lambda$, $\eta$, and $\kappa$ is defined by

$$\pi(\lambda,\ \eta,\ \kappa) = \frac{e^{-(\zeta_1\lambda+\zeta_2\eta+\zeta_3\kappa)}\ {\zeta_1}^{\nu_1}\ {\zeta_2}^{\nu_2}\ {\zeta_3}^{\nu_3}}{\Gamma(\nu_1)\ \Gamma(\nu_2)\ \Gamma(\nu_3)}\ \lambda^{\nu_1-1}\ \eta^{\nu_2-1}\ \kappa^{\nu_3-1};\qquad \lambda>0,\ \eta>0,\ \kappa>0,\ (\zeta_1,\ \nu_1,\ \zeta_2,\ \nu_2,\ \zeta_3,\ \nu_3>0). \tag{3.1}$$

By using equations (2.1) and (3.1) we get the posterior distribution of $\lambda,\ \eta$ and $\kappa$ as follows

$$\pi\left(\lambda,\ \eta,\ \kappa \mid \underline{y}\right) = \frac{\alpha\ \beta}{\int_0^\infty\int_0^\infty\int_0^\infty \alpha\ \beta\ d\lambda\ d\eta\ d\kappa}, \tag{3.2}$$

where

$$\alpha = e^{-(\zeta_1\lambda+\zeta_2\eta+\zeta_3\kappa)}\ \lambda^{\nu_1+m-1}\ \eta^{\nu_2+m-1}\ \kappa^{\nu_3+m-1}\ \prod_{i=1}^{m} y_i^{\kappa-1}(1+{y_i}^{\kappa})^{-(\lambda+1)},$$

and

$$\beta = \prod_{i=1}^{m}\left[1-(1+{y_i}^{\kappa})^{-\lambda}\right]^{\eta-1}\left(1-\left[1-(1+{y_i}^{\kappa})^{-\eta}\right]^{\lambda}\right)^{(k(R_i+1)-1)}.$$

Integration in the equation (3.2) cannot be obtained in a closed form, so we solve it numerically. In the following subsections we derive Bayesian estimators for the parameters $\lambda,\ \eta,\ \kappa$, the reliability, hazard rate, and reversed hazard rate functions under different loss functions.

### 3.1. Bayesian Estimators Under Square Error Loss Function

(1) The Bayesian estimator of the parameter $\lambda$ is given by

$$\hat{\lambda}_{sq} = E(\lambda) = \int_0^\infty \int_0^\infty \int_0^\infty \left(\lambda \, \pi \left(\lambda, \ \eta, \ \kappa \mid \underline{y}\right)\right) \, d\lambda \, d\eta \, d\kappa,$$

provided that $E(\lambda)$ exists. This integration cannot be solved analytically, so we use the Lindley's Bayes approximation for any function $\psi$ of parameter $\omega$, $\omega = (\theta_1, \ \theta_2, \ \theta_3)$ and $Q(\theta_1, \ \theta_2, \ \theta_3) = \log \pi(\theta_1, \ \theta_2, \ \theta_3)$, which is defined by

$$E\left(\psi(\omega) \mid \underline{y}\right) \approx \Bigg(\psi(\theta_1, \ \theta_2, \ \theta_3) + \frac{1}{2}\Bigg[\sum_r \sum_s \left(\psi_{rs} + 2\psi_r Q_s\right)\sigma_{rs}$$

$$+ \sum_r \sum_s \sum_z \sum_w L_{rsz}\psi_w \sigma_{rs}\sigma_{zw}\Bigg]\Bigg)_{(\theta_1, \ \theta_2, \ \theta_3)_{ML}}, \ \forall \ r, \ s, \ z, \ w = 1, \ 2, \ 3, \quad (3.3)$$

where $Q_i = \frac{\partial Q(\theta_1, \ \theta_2, \ \theta_3)}{\partial \theta_i}$, $\psi_i = \frac{\partial \psi(\theta_1, \ \theta_2, \ \theta_3)}{\partial \theta_i}$, $\psi_{ij} = \frac{\partial^2 \psi(\theta_1, \ \theta_2, \ \theta_3)}{\partial \theta_i \ \partial \theta_j}$, $L_{ij} = \frac{\partial^2 \ell}{\partial \theta_i \ \partial \theta_j}$, $L_{ijk} = \frac{\partial^3 \ell}{\partial \theta_i \ \partial \theta_j \ \partial \theta_k}$, $\forall \ i, \ j, \ k = 1, \ 2, \ 3$, and $\sigma_{ij} = (i,j)^{th}$ element in the matrix

$$\begin{pmatrix} -L_{11} & -L_{12} & -L_{13} \\ -L_{21} & -L_{22} & -L_{23} \\ -L_{31} & -L_{32} & -L_{33} \end{pmatrix}^{-1}, \forall \ i, \ j = 1, \ 2, \ 3.$$

Substitute in the equation (3.3), $\psi = \lambda$, the Bayesian estimator of the shape parameter $\lambda$ is given as

$$\hat{\lambda}_{sq} \approx \lambda + Q_1\sigma_{11} + Q_2\sigma_{12} + Q_3\sigma_{13} + \frac{1}{2}\left(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}\right),$$

where

$$A = \sigma_{11}L_{111} + \sigma_{22}L_{221} + \sigma_{33}L_{331} + 2\left(\sigma_{12}L_{121} + \sigma_{13}L_{131} + \sigma_{23}L_{231}\right),$$
$$B = \sigma_{11}L_{112} + \sigma_{22}L_{222} + \sigma_{33}L_{332} + 2\left(\sigma_{12}L_{122} + \sigma_{13}L_{132} + \sigma_{23}L_{232}\right),$$
$$C = \sigma_{11}L_{113} + \sigma_{22}L_{223} + \sigma_{33}L_{333} + 2\left(\sigma_{12}L_{123} + \sigma_{13}L_{133} + \sigma_{23}L_{233}\right).$$

(2) Substitute in the equation (3.3), $\psi = \eta$, the Bayesian estimator of the parameter $\eta$ is given as

$$\hat{\eta}_{sq} \approx \left(\eta + Q_1\sigma_{21} + Q_2\sigma_{22} + Q_3\sigma_{23} + \frac{1}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)\right).$$

(3) Substitute in the equation (3.3), $\psi = \kappa$, the Bayesian estimator of the parameter

$\kappa$ is given as

$$\hat{\kappa}_{sq} \approx \left(\kappa + Q_1\sigma_{31} + Q_2\sigma_{32} + Q_3\sigma_{33} + \frac{1}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right).$$

(4) Substitute in the equation (3.3), $\psi = R\left(t\right)$, the Bayesian estimator of the reliability function $R(t)$ is given as

$$\hat{R}_{sq}\left(t\right) \approx R\left(t\right) + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right) + \frac{1}{2}[A\left(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}\right)$$

$$+B\left(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}\right) + C\left(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}\right)],$$

where

$$\begin{aligned} a_i &= Q_1\sigma_{i1} + Q_2\sigma_{i2} + Q_3\sigma_{i3}; i = 1, 2, 3, \\ a_4 &= \psi_{12}\sigma_{12} + \psi_{13}\sigma_{13} + \psi_{23}\sigma_{23}, \\ a_5 &= \frac{1}{2}\left(\psi_{11}\sigma_{11} + \psi_{22}\sigma_{22} + \psi_{33}\sigma_{33}\right). \end{aligned}$$

(5) Substitute in the equation (3.3), $\psi = H\left(t\right)$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{sq}\left(t\right) \approx H\left(t\right) + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right) + \frac{1}{2}[A\left(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}\right)]$$

$$+B\left(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}\right) + C\left(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}\right).$$

(6) Substitute in the equation (3.3), $\psi = H^{\star}\left(t\right)$, the Bayesian estimator of the reversed hazard rate function $H^{\star}(t)$ is given by

$$\hat{H}^{\star}_{sq}\left(t\right) \approx H^{\star}\left(t\right) + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right) + \frac{1}{2}[A\left(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}\right)$$

$$+B\left(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}\right) + C\left(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}\right)].$$

### 3.2. Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

(1) Substitute in the equation (3.3), $\psi = e^{-\rho\lambda}$, the Bayesian estimator of the shape parameter $\lambda$ is given as

$$\hat{\lambda}_{LINEX} \approx -\frac{1}{\rho}\log\left[e^{-\rho\lambda} - \frac{\rho}{e^{\lambda\rho}}\left(Q_1\sigma_{11} + Q_2\sigma_{12} + Q_3\sigma_{13}\right) + \frac{\rho^2}{2e^{\lambda\rho}}\sigma_{11} - \frac{\rho}{2e^{\lambda\rho}}\left(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}\right)\right].$$

(2) The Bayesian estimator of the parameter $\eta$ is given by

$$\hat{\eta}_{LINEX} = -\frac{1}{\rho}\log\left[E\left(e^{-\rho\eta}\right)\right],$$

provided that $E\left(e^{-\rho\eta}\right)$ exists. Substitute in the equation (3.3), $\psi = e^{-\rho\eta}$, the Bayesian estimator of the parameter $\eta$ is given by

$$\hat{\eta}_{LINEX} \approx -\tfrac{1}{\rho}\log\left[e^{-\rho\eta} - \tfrac{\rho}{e^{\eta\rho}}\left(Q_1\sigma_{21} + Q_2\sigma_{22} + Q_3\sigma_{23}\right) + \tfrac{\rho^2}{2e^{\eta\rho}}\sigma_{22} - \tfrac{\rho}{2e^{\eta\rho}}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)\right].$$

(3) Substitute in the equation (3.3), $\psi = e^{-\rho\kappa}$, the Bayesian estimator of the shape parameter $\kappa$ is given by

$$\hat{\kappa}_{LINEX} \approx -\tfrac{1}{\rho}\log\left[e^{-\rho\kappa} - \tfrac{\rho}{e^{\kappa\rho}}\left(Q_1\sigma_{31} + Q_2\sigma_{32} + Q_3\sigma_{33}\right) + \tfrac{\rho^2}{2e^{\kappa\rho}}\sigma_{33} - \tfrac{\rho}{2e^{\kappa\rho}}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right].$$

(4) Substitute in the equation (3.3), $\psi = e^{-\rho R(t)}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{LINEX}(t) \approx -\frac{1}{\rho}\log\Big[e^{-\rho R(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)$$

$$+\frac{1}{2}[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12} + \psi_3\sigma_{13}\right) + B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right)]\Big].$$

(5) Substitute in the equation (3.3), $\psi = e^{-\rho H(t)}$, the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{LINEX}(t) \approx -\frac{1}{\rho}\log\Big[e^{-\rho H(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)$$

$$+\frac{1}{2}\left[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}\right)+B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right)\right]\Big].$$

(6) Substitute in the equation (3.3), $\psi = e^{-\rho H^\star(t)}$, the Bayesian estimator of the reversed hazard rate function $H^\star(t)$ is given as

$$\hat{H}^\star_{LINEX}(t) \approx -\frac{1}{\rho}\log\Big[e^{-\rho H^\star(t)} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right)$$

$$+\frac{1}{2}[A\left(\psi_1\sigma_{11}+\psi_2\sigma_{12}+\psi_3\sigma_{13}\right)+B\left(\psi_1\sigma_{21}+\psi_2\sigma_{22}+\psi_3\sigma_{23}\right)+C\left(\psi_1\sigma_{31}+\psi_2\sigma_{32}+\psi_3\sigma_{33}\right)]\Big].$$

### *3.3. Bayesian Estimators Under General Entropy Loss Function*

(1) The Bayesian estimator of the shape parameter $\lambda$ is given by

$$\hat{\lambda}_{Gentropy} = \left[E\left(\lambda^{-q}\right)\right]^{\frac{-1}{q}},$$

provided that $E\left(\lambda^{-q}\right)$ exists. Substitute in the equation (3.3), $\psi = \lambda^{-q}$, the Bayesian estimator of the parameter $\lambda$ is given by

$$\hat{\lambda}_{\text{Gentropy}} \approx \begin{bmatrix} \lambda^{-q} - q\lambda^{-(q+1)}\left(Q_1\sigma_{11} + Q_2\sigma_{12} + Q_3\sigma_{13}\right) + \frac{\left((q+1)\,q\lambda^{-(q+2)}\right)}{2}\sigma_{11} \\ -\frac{q\lambda^{-(q+1)}}{2}\left(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

(2) Substitute in the equation (3.3), $\psi = \eta^{-q}$, the Bayesian estimator of the shape parameter $\eta$ is given by

$$\hat{\eta}_{\text{Gentropy}} \approx \begin{bmatrix} \eta^{-q} - q\eta^{-(q+1)}\left(Q_1\sigma_{21} + Q_2\sigma_{22} + Q_3\sigma_{23}\right) + \frac{\left((q+1)\,q\eta^{-(q+2)}\right)}{2}\sigma_{22} \\ -\frac{q\eta^{-(q+1)}}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

(3) Substitute in the equation (3.3), $\psi = \kappa^{-q}$, the Bayesian estimator of the shape parameter $\kappa$ is given by

$$\hat{\kappa}_{\text{Gentropy}} \approx \begin{bmatrix} \kappa^{-q} - q\kappa^{-(q+1)}\left(Q_1\sigma_{31} + Q_2\sigma_{32} + Q_3\sigma_{33}\right) + \frac{\left((q+1)\,q\kappa^{-(q+2)}\right)}{2}\sigma_{33} \\ -\frac{q\kappa^{-(q+1)}}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right) \end{bmatrix}^{\frac{-1}{q}}.$$

(4) Substitute in the equation (3.3), $\psi = R(t)^{-q}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy}(t) \approx \left[ (R(t))^{-q} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right) + \frac{1}{2}\begin{bmatrix} A\left(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}\right) \\ +B\left(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}\right) \\ +C\left(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}\right) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

(5) Substitute in the equation (3.3), $\psi = H(t)^{-q}$, the Bayesian estimator of the hazard rate function $H(t)$ is given as

$$\hat{H}_{Gentropy}(t) \approx \left[ (H(t))^{-q} + \left(\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5\right) + \frac{1}{2}\begin{bmatrix} A\left(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}\right) \\ +B\left(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}\right) \\ +C\left(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}\right) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

(6) Substitute in the equation (3.3), $\psi = H^{\star}(t)^{-q}$, the Bayesian estimator of the

reversed hazard rate function $H^{\star}(t)$ is given as

$$\hat{H}^{\star}_{Gentropy}(t) \approx \left[ (H^{\star}(t))^{-q} + (\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 + a_4 + a_5) + \frac{1}{2} \begin{bmatrix} A(\psi_1\sigma_{11} + \psi_2\sigma_{12} + \psi_3\sigma_{13}) \\ +B(\psi_1\sigma_{21} + \psi_2\sigma_{22} + \psi_3\sigma_{23}) \\ +C(\psi_1\sigma_{31} + \psi_2\sigma_{32} + \psi_3\sigma_{33}) \end{bmatrix} \right]^{\frac{-1}{q}}.$$

## 4. Simulation studies

In this section, we conduct a Monte Carlo simulation study to compare the performance of various estimates developed in the previous sections. A large number (10000) of progressive first-failure censored samples are generated from the model (2.1). These generated samples are of varying group sizes $k = 3;\ 6$; number of groups in a sample $n = 50;\ 80$ and effective sample sizes $m = 25;\ 40$ out of $n$ and different combinations of progressive censoring schemes $\underline{R}$. The study includes the following steps:

(1) Generate a progressive first-failure censored sample using algorithm proposed by Balakrishnan and Sandhu (1995) from model (2.1) for given values of $(k, n, m, \underline{R})$.
(2) Calculate the maximum likelihood estimates of $\lambda,\ \eta,\ \kappa,\ R(t),\ H(t)$, and $H^{\star}(t)$. according to Section 2.
(3) According to Section 3, obtain the Bayes estimates of $\lambda,\ \eta,\ \kappa,\ R(t),\ H(t)$, and $H^{\star}(t)$.
(4) Repeat the steps $(1) - (3)$, (10000) times, for different values of $(k, n, m, \underline{R})$.

We consider the estimation average $= \frac{\sum_{i=1}^{10000} \hat{\theta}_i}{10000}$ and the mean square error $= \frac{\sum_{i=1}^{10000} \left(\hat{\theta}_i - \theta\right)^2}{10000}$, where $\theta$ is the parameter and $\hat{\theta}$ is its estimator. Extensive computations are performed using Mathematica 11. Note that, since the non-linear equations (2.3) are not solvable analytically, numerical methods can be used, as Newton Raphson method with initial values closed to real values of the parameters.
Throughout this section we will use the following abbreviations:

(1) $MSEs$ : The mean square errors,
(2) $ML$ : The estimate by using the MLE,
(3) $B_{Sq}$ : The estimate under squared error loss function,
(4) $B_{Lx,c=3}$: The estimate under linex loss function at $c = 3$,
(5) $B_{Lx,c=6}$: The estimate under linex loss function at $c = 6$,
(6) $B_{Ge,q=4}$: The estimate under general entropy loss function at $q = 4$,
(7) $B_{Ge,q=8}$: The estimate under general entropy loss function at $q = 8$.

**Table 1.** Average values of the estimates and the corresponding MSEs, given in parentheses of the parameters $\lambda,\ \eta,\ \kappa$; when $\lambda = 1.2,\ \eta = 0.7,\ \kappa = 0.9,\ \zeta_1 = 2,\ \nu_1 = 3,\ \zeta_2 = 2,\ \nu_2 = 3$ and $\zeta_3 = 2,\ \nu_3 = 3$

| $B_{Lx,c=3}$ | $B_{Lx,c=6}$ | $B_{Ge,q=4}$ | $B_{Ge,q=8}$ | $B_{Sq}$ | $ML$ | Scheme | $(k,n,m)$ |
|---|---|---|---|---|---|---|---|
| *The average estimates of $\lambda$ (attached with the MSEs)* | | | | | | | |
| 1.35943 | 1.68111 | 1.27847 | 1.19889 | 1.31227 | 1.25972 | (10,18*0,10) | (3,40,20) |
| (0.02185) | (0.18061) | (0.00439) | (0.00017) | (0.00003) | (0.05447) | | |
| 1.30941 | 1.4565 | 1.2571 | 1.2241 | 1.2730 | 1.2469 | (20,19*0) | |
| (0.02141) | (0.17681) | (0.02451) | (0.00261) | (0.00004) | (0.04381) | | |
| 1.40031 | 1.27794 | 1.25570 | 1.23664 | 1.26891 | 1.22818 | (19*0,20) | |
| (0.03641) | (0.02140) | (0.00192) | (0.00060) | (0.00005) | (0.04018) | | |
| 1.32198 | 1.37245 | 1.25392 | 1.2263 | 1.26055 | 1.23427 | (20,38*0,20) | (6,80,40) |
| (0.01207) | (0.03721) | (0.00174) | (0.00020) | (0.00006) | (0.02233) | | |
| 1.29566 | 1.37622 | 1.25811 | 1.22871 | 1.27265 | 1.24206 | (40,39*0) | |
| (0.00696) | (0.02725) | (0.00210) | (0.00271) | (0.00007) | (0.02095) | | |
| 1.3479 | 1.39251 | 1.29431 | 1.22994 | 1.2841 | 1.25741 | (39*0,40) | |
| (0.02511) | (0.04371) | (0.00579) | (0.00489) | (0.00893) | (0.03192) | | |
| *The average estimates of $\eta$ (attached with the MSEs)* | | | | | | | |
| 0.76091 | 0.75498 | 0.73010 | 0.71703 | 0.77172 | 0.73353 | (10,18*0,10) | (3,40,20) |
| (0.00291) | (0.002281) | (0.00053) | (0.00011) | (0.00419) | (0.01630) | | |
| 0.75341 | 0.74312 | 0.72110 | 0.71320 | 0.75184 | 0.72161 | (20,19*0) | |
| (0.00187) | (0.002270) | (0.00413) | (0.00012) | (0.00325) | (0.01420) | | |
| 0.76737 | 0.76835 | 0.73024 | 0.72221 | 0.75141 | 0.72486 | (19*0,20) | |
| (0.000364) | (0.00375) | (0.00054) | (0.00023) | (0.00196) | (0.02153) | | |
| 0.74138 | 0.73562 | 0.72330 | 0.70887 | 0.73371 | 0.71713 | (20,38*0,20) | (6,80,40) |
| (0.00117) | (0.00081) | (0.00026) | (0.00042) | (0.00071) | (0.00979) | | |
| 0.75275 | 0.72147 | 0.73750 | 0.69853 | 0.75753 | 0.73562 | (40,39*0) | |
| (0.00207) | (0.00020) | (0.00092) | (0.00007) | (0.00253) | (0.00989) | | |
| 0.75481 | 0.73025 | 0.73121 | 0.70817 | 0.75042 | 0.72994 | (39*0,40) | |
| (0.00317) | (0.00031) | (0.00081) | (0.00067) | (0.00194) | (0.00874) | | |
| *The average estimates of $\kappa$ (attached with the MSEs)* | | | | | | | |
| 1.00307 | 0.92720 | 0.97195 | 0.89376 | 1.04754 | 0.98601 | (10,18*0,10) | (3,40,20) |
| (0.00878) | (0.00032) | (0.00392) | (0.00078) | (0.01912) | (0.03760) | | |
| 0.96402 | 0.92511 | 0.95410 | 0.90123 | 0.99841 | 0.94261 | (20,19*0) | |
| (0.00671) | (0.00031) | (0.00291) | (0.00054) | (0.00942) | (0.02541) | | |
| 0.99701 | 0.92189 | 0.95800 | 0.88004 | 1.00548 | 0.95995 | (19*0,20) | |
| (0.00770) | (0.00016) | (0.00238) | (0.00083) | (0.00925) | (0.03975) | | |
| 0.96374 | 0.93341 | 0.94441 | 0.90890 | 0.96633 | 0.94035 | (20,38*0,20) | (6,80,40) |
| (0.00297) | (0.00059) | (0.00124) | (0.00001) | (0.00325) | (0.01567) | | |
| 0.95893 | 0.91475 | 0.94267 | 0.89346 | 0.97462 | 0.94319 | (40,39*0) | |
| (0.00247) | (0.00003) | (0.00112) | (0.00024) | (0.00427) | (0.01734) | | |
| 0.96942 | 0.93682 | 0.95320 | 0.91862 | 0.97642 | 0.95320 | (39*0,40) | |
| (0.00346) | (0.00042) | (0.00294) | (0.00943) | (0.00541) | (0.02412) | | |

From Tables 1 and 2, we observe that the $MLE$ and Bayes estimates of the parameters $\lambda,\ \eta,\ \kappa$, the reliability, hazard rate, and reversed hazard rate functions are very good in terms of $MSEs$. As the number of groups $n$ and effective sample size $m$ increase, $MSEs$ of all estimates decrease as expected. Also, as the value of the group size $k$ increases, $MSEs$ decrease. In general, the Bayesian estimators have $MSEs$ less than that of the $MlE$. Bayes estimates using gamma informative prior are better as they include prior information than $MLE$ in terms of $MSEs$.

## 5. Conclusion

In this paper, assuming a good lifetime model we consider the problem of estimating the unknown parameters $\lambda,\ \eta,\ \kappa$, as well as the reliability, hazard rate, and reversed

**Table 2.** Average values of the estimates and the corresponding MSEs, given in parentheses of the relaibility, hazard rate and reversed hazard rate functions; when $\lambda = 1.2,\ \eta = 0.7,\ \kappa = 0.9,\ \zeta_1 = 2,\ \nu_1 = 3,\ \zeta_2 = 2,\ \nu_2 = 3$ and $\zeta_3 = 2,\ \nu_3 = 3$

| $B_{Lx,c=3}$ | $B_{Lx,c=6}$ | $B_{Ge,q=4}$ | $B_{Ge,q=8}$ | $B_{Sq}$ | $ML$ | Scheme | $(k,n,m)$ |
|---|---|---|---|---|---|---|---|
| *The average estimates of relaibility function* R(t=2)=0.166423 (*attached with the MSEs*) | | | | | | | |
| 0.17912 | 0.16941 | 0.16720 | 0.16689 | 0.17941 | 0.16748 | (10,18*0,10) | (3,40,20) |
| (0.00214) | (0.00211) | (0.00203) | (0.00041) | (0.00433) | (0.00361) | | |
| 0.16543 | 0.16698 | 0.16842 | 0.16659 | 0.16942 | 0.16710 | (20,19*0) | |
| (0.00147) | (0.00741) | (0.00427) | (0.00864) | (0.00124) | (0.02612) | | |
| 0.16843 | 0.16979 | 0.16851 | 0.16791 | 0.17241 | 0.16251 | (19*0,20) | |
| (0.01540) | (0.00857) | (0.07024) | (0.00135) | (0.00752) | (0.04215) | | |
| 0.16743 | 0.16681 | 0.16654 | 0.16632 | 0.16942 | 0.16695 | (20,38*0,20) | (6,80,40) |
| (0.00324) | (0.00331) | (0.00421) | (0.00362) | (0.00446) | (0.00652) | | |
| 0.16841 | 0.16645 | 0.16643 | 0.166394 | 0.166871 | 0.16742 | (40,39*0) | |
| (0.00126) | (0.00011) | (0.00034) | (0.00056) | (0.00157) | (0.00872) | | |
| 0.16794 | 0.16710 | 0.16842 | 0.16857 | 0.16998 | 0.16773 | (39*0,40) | |
| (0.09471) | (0.08110) | (0.00573) | (0.00446) | (0.00841) | (0.00492) | | |
| *The average estimates of hazard rate function* H(t=2)=0.427115 (*attached with the MSEs*) | | | | | | | |
| 0.49871 | 0.49431 | 0.54468 | 0.49937 | 0.50241 | 0.46942 | (10,18*0,10) | (3,40,20) |
| (0.00762) | (0.00834) | (0.00091) | (0.00432) | (0.00230) | (0.01342) | | |
| 0.46422 | 0.45873 | 0.45991 | 0.48231 | 0.49701 | 0.45881 | (20,19*0) | |
| (0.00696) | (0.02725) | (0.00027) | (0.00295) | (0.00121) | (0.00298) | | |
| 0.54213 | 0.48681 | 0.46841 | 0.47332 | 0.49941 | 0.47814 | (19*0,20) | |
| (0.00624) | (0.00513) | (0.00134) | (0.00298) | (0.02567) | (0.00247) | | |
| 0.46834 | 0.48781 | 0.46284 | 0.47684 | 0.49203 | 0.44682 | (20,38*0,20) | (6,80,40) |
| (0.03801) | (0.00078) | (0.00321) | (0.00878) | (0.02461) | (0.01325) | | |
| 0.45256 | 0.46866 | 0.42644 | 0.42948 | 0.45791 | 0.43245 | (40,39*0) | |
| (0.00003) | (0.00004) | (0.00002) | (0.00002) | (0.00045) | (0.03221) | | |
| 0.47891 | 0.48976 | 0.47792 | 0.46987 | 0.49921 | 0.45689 | (39*0,40) | |
| (0.00269) | (0.00094) | (0.00243) | (0.00392) | (0.03760) | (0.00475) | | |
| *The average estimates of reversed hazard rate function* H*(t=2)=0.0852734 (*with the MSEs*) | | | | | | | |
| 0.1289 | 0.10941 | 0.19987 | 0.09872 | 0.0241 | 0.10993 | (10,18*0,10) | (3,40,20) |
| (0.02968) | (0.09823) | (0.09948) | (0.06289) | (0.06819) | (0.04871) | | |
| 0.0384 | 0.04891 | 0.05979 | 0.06871 | 0.06681 | 0.05481 | (20,19*0) | |
| (0.000841) | (0.00712) | (0.00324) | (0.07862) | (0.03421) | (0.00524) | | |
| 0.13874 | 0.11874 | 0.09987 | 0.05481 | 0.03394 | 0.10024 | (19*0,20) | |
| (0.02311) | (0.02461) | (0.01243) | (0.00942) | (0.01265) | (0.09461) | | |
| 0.09948 | 0.08947 | 0.07689 | 0.07681 | 0.06814 | 0.09841 | (20,38*0,20) | (6,80,40) |
| (0.00245) | (0.07077) | (0.05947) | (0.00397) | (0.00869) | (0.00620) | | |
| 0.08923 | 0.09321 | 0.085241 | 0.085121 | 0.08687 | 0.08873 | (40,39*0) | |
| (0.00004) | (0.00871) | (0.00001) | (0.00001) | (0.00022) | (0.00364) | | |
| 0.08894 | 0.09861 | 0.09783 | 0.08932 | 0.07812 | 0.08931 | (39*0,40) | |
| (0.00254) | (0.01540) | (0.00528) | (0.00542) | (0.00342) | (0.00184) | | |

hazard rate functions using progressive first-failure censored samples. This censoring scheme has advantages in terms of reducing test time, in which more items are used but only $m$ of $(k \times n)$ items failed. We derived MLE and Bayes estimators of the parameters $\lambda,\ \eta,\ \kappa$, the reliability, hazard rate, and reversed hazard rate functions using gamma informative priors, under both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. These estimates cannot be obtained in closed form, but can be computed numerically. It is clear that the proposed Bayes estimators perform very well for different $n$ and $m$. Also the Bayes estimators based on gamma informative priors perform much better than the $MLE$ in terms of $MSEs$. The simulation also stresses the importance of linex and general entropy loss functions as asymmetric loss functions, in the case studied.

**Acknowledgements**

The authors are extremely grateful to the editor and the referees for their valuable suggestions which helped in improving the manuscript.

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